Chapter 5

PORTFOLIO SELECTION IN THE PRESENCE OF HEAVY–TAILED ASSET RETURNS*

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Abstract We discuss the question of portfolio selection when the returns of the assets under consideration are characterized by a heavy–tailed distribution. As distributional assumption we consider the sub–Gaussian stable model and address the problems of estimation and portfolio optimization. The advantages for risk assessment when relaxing the normal assumption in favor of the heavy–tailed variant are illustrated empirically.

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1. Introduction

Although the normal distribution is almost a "universal law" for random phenomena encountered in nature, but also in engineering and social systems, there are a number of well–know phenomena whose distribution is heavier tailed than the normal model implies. The stable Paretian distribution, which represents a generalization of the normal, arises as a natural candidate for such cases. Stable distributions accommodate heavy tails and skewness while still preserving desirable properties of the normal distribution. One such property is that stable distributions have a domain of attraction, implying that any distribution in the domain of attraction of a specific stable distribution will have properties that are close to those of this stable distribution. Another desirable feature of the stable model is the stability property; that is, stable distributions are closed under summation of independent and identically distributed stable random variables.

Practical applications of the stable distribution can be found in physics, engineering and signal processing (see, for example, Nikias and Shao, 1995), but they also occur in statistical inference. Hansen, Kim and Mittnik (1998) propose a modified χ^2–test for cointegrating coefficients which takes the heavy–tailedness of the finite–sample distribution explicitly into account. However, it is the field of finance where the stable model has achieved particular prominence. The fundamental work of Mandelbrot (1963a,b, 1967) and Fama (1963, 1965a,b) initiated the interest in studying the empirical distribution of financial assets. The excess kurtosis, which Mandelbrot and Fama found in their empirical analyses, led them to reject the normal assumption and to propose the heavy–tailed stable Paretian distribution as a more realistic description of asset return behavior.[1]

In the following, we are concerned with the portfolio selection problem. Specifically, we consider the problem of constructing an optimal portfolio from n assets which follow a particular variant of a stable model, namely, a joint sub-Gaussian stable distribution. The fact that this model belongs to the class of elliptical distributions allows us to use a mean–dispersion approach to portfolio selection—a generalization of Markowitz's (1959) mean–variance approach (see Owen and Rabinovitch, 1983). In order to accommodate asymmetric return distributions, we then review the three–fund separation model for returns in the domain of attraction of a stable law (see Ortobelli, Rachev and Schwartz, 2000; Ortobelli, Huber, Rachev and Schwartz, 2001; and Rachev, Ortobelli and Huber, 2001). In the asymmetric case, the model results from a stable version of Simaan's (1993) model. The symmetric case resembles a model studied by Götzenberger, Rachev and Schwartz (2000) which can be viewed as a particular version of the two–fund separation put forth by Fama (1965b).

Finally, we are interested in comparing the normal and stable models with respect to their capability of correctly determining the risk of a portfolio. Focusing on the value–at–risk (VaR) measure for risk assessment, we look into the empirical accuracy resulting from these two assumption and report on back-testing exercises for a portfolio of stocks belonging to the German DAX index.

2. The Sub-Gaussian Stable Model

2.1 Portfolio Choice

Let r_i, $i = 1, \dots n$, denote the return on risky asset i and assume that the vector of returns, $r = [r_1, \dots, r_n]'$ follows a sub–Gaussian stable distribution with characteristic function

$$\Phi_r(t) = \mathrm{E}\exp(it'r) = \exp\left\{-\left(t'Qt\right)^{\alpha/2} + it'\mu\right\}, \tag{5.1}$$

where, assuming the stable exponent satisfies $1 < \alpha < 2$, $\mu = \mathrm{E}(r)$ is the mean vector; and $Q = [R_{i,j}/2]_{ij}$ is a positive definite $n \times n$ matrix. The term $R_{i,j}$ is defined by

$$\frac{R_{i,j}}{2} = [\widetilde{r}_i, \widetilde{r}_j]_\alpha \, \|\widetilde{r}_j\|_\alpha^{2-\alpha}, \tag{5.2}$$

where $\widetilde{r}_j = r_j - \mu_j$ denotes the de–meaned return, and $[\widetilde{r}_i, \widetilde{r}_j]_\alpha$ is the covariation between the two jointly symmetric stable random variables $\widetilde{r}_i$ and $\widetilde{r}_j$ which is given by

$$[\widetilde{r}_i, \widetilde{r}_j]_\alpha = \int_{S_2} s_i \, |s_j|^{\alpha-1} \operatorname{sgn}(s_j)\gamma(ds).$$

Here, $\gamma(ds)$ is the spectral measure with support on the n–dimensional unit sphere, $S_n = \{s \in \mathbb{R}^n \mid \|s\| = 1\}$. Moreover,

$$\|\widetilde{r}_j\|_\alpha = \left(\int_{S_2} |s_j|^\alpha \, \gamma(ds)\right)^{1/\alpha} = \left([\widetilde{r}_j, \widetilde{r}_j]_\alpha\right)^{1/\alpha},$$

where the spectral measure, $\gamma(ds)$, has support on the unit circle, S_2.

The sub–Gaussian stable model can be considered as a special case of the elliptical model of Owen and Rabinovitch (1983). To estimate the efficient–portfolio frontier from observed return data, Rachev, Ortobelli and Huber (2001) rely on the sample mean as an estimate for vector μ and, to obtain $\hat{Q}$, use Lemma 2.7.16 in Samorodnitsky and Taqqu (1994), according to which, for every p with $1 < p < \alpha$,

$$\frac{[\widetilde{r}_i, \widetilde{r}_j]_\alpha}{\|\widetilde{r}_j\|_\alpha^\alpha} = \frac{\mathrm{E}\left(\widetilde{r}_i \widetilde{r}_j^{\langle p-1\rangle}\right)}{\mathrm{E}\left(|\widetilde{r}_j|^p\right)}, \tag{5.3}$$

where $x^{<a>} = \operatorname{sgn}(x)|x|^a$ and the scale parameter, σ_j, is written as $\|\widetilde{r}_j\|_\alpha = \sigma_j$. In the symmetric case, Property 1.2.17 in Samorodnitsky and Taqqu (1994) implies that

$$\sigma_j^p = \|\widetilde{r}_j\|_\alpha^p = \frac{\mathrm{E}\left(|r_j - \mu_j|^p\right) p \displaystyle\int_0^{+\infty} u^{-p-1} \sin^2 u du}{2^{p-1}\Gamma\left(1 - \frac{p}{\alpha}\right)}\ , \tag{5.4}$$

so that

$$\frac{R_{i,j}}{2} = \sigma_j^2 \frac{\mathrm{E}\left(\widetilde{r}_i \widetilde{r}_j^{\langle p-1\rangle}\right)}{\mathrm{E}\left(|\widetilde{r}_j|^p\right)}.$$

Then, an estimate of the typical element, $\hat{q}_{ij} = \hat{R}_{ij}/2$, of matrix Q is given by

$$\frac{\widehat{R}_{i,j}}{2} = \widehat{\sigma}_j^2 \frac{\displaystyle\sum_{k=1}^{N} \widetilde{r}_i^{(k)} \left(\widetilde{r}_j^{(k)}\right)^{\langle p-1\rangle}}{\displaystyle\sum_{k=1}^{N} \left|\widetilde{r}_j^{(k)}\right|^p}, \tag{5.5}$$

with

$$\widehat{\sigma}_j^2 = \frac{\widehat{R}_{j,j}}{2} = \left(\frac{\frac{1}{N} \displaystyle\sum_{k=1}^{N} \left|\widetilde{r}_j^{(k)}\right|^p p \int_0^{+\infty} u^{-p-1} \sin^2 u du}{2^{p-1}\Gamma\left(1 - \frac{p}{\alpha}\right)} \right)^{2/p} \tag{5.6}$$

being an estimate of σ_j^2. Letting $x = [x_1, \ldots, x_n]'$ be the $n \times 1$ vector of portfolio weights associated with the n risky assets, then, for the portfolio return on the risky assets, $x'r$, we have

$$x'r \stackrel{d}{=} S_\alpha(\sigma_{x'r}, \beta_{x'r}, m_{x'r}).$$

In the presence of a risk–free asset with certain return z_0, the overall portfolio return, denoted by P, is

$$P = \begin{cases} z_0, & \text{if} \quad x = 0, \\ x'r + (1 - x'\mathbf{1})z_0, & \text{otherwise.} \end{cases}$$

Then, for $x \neq 0$,

$$P = x'r + (1 - x'\mathbf{1})z_0 \stackrel{d}{=} S_\alpha(\sigma_{x'r}, \beta_{x'r}, m_P),$$

where **1** denotes an $n \times 1$ vector of ones; $\sigma_{x'r} = \sqrt{x'Qx}$ is the scale parameter of the risky–portfolio return; $\beta_{x'r} = 0$ is the skewness parameter; and, finally,

$m_P = x'\mathrm{E}(r) + (1 - x'\mathbf{1})z_0$ denotes the mean of the return on the overall portfolio.

If the return vector is jointly sub–Gaussian stable, risk–averse investors will choose an optimal portfolio by solving the optimization problem

$$\begin{aligned} &\min_x x'Qx \\ \text{subject to} \quad &x'\mu + (1 - x'\mathbf{1})z_0 = m_P. \end{aligned} \tag{5.7}$$

Thus, every optimal portfolio that maximizes a given concave utility function, U, i.e.,

$$\max_x \mathrm{E}\left[U(x'r + (1 - x'\mathbf{1})z_0)\right],$$

belongs to the mean-dispersion frontier with scale

$$\sigma = \frac{|m - z_0|}{\sqrt{(\mu - \mathbf{1}z_0)'Q^{-1}(\mu - \mathbf{1}z_0)}}, \tag{5.8}$$

where $m = x'\mu + (1 - x'\mathbf{1})z_0$ and $\sigma^2 = x'Qx$. The optimal portfolio weights satisfy

$$x = \frac{m - z_0}{(\mu - \mathbf{1}z_0)'Q^{-1}(\mu - \mathbf{1}z_0)} Q^{-1}(\mu - z_0\mathbf{1}). \tag{5.9}$$

Note that expressions (5.8) and (5.9) resemble those of the traditional mean-variance approach. However, even if Q is a symmetric matrix—it is definite positive—the estimator based on (5.5) and (5.6) will generally not be symmetric in the sub-Gaussian cases. Hence, in applied work it may happen that the estimate of the squared scale parameter for the risky portfolio, $x'r$, is negative[2].

Although expression (5.9) exhibits the two–fund–separation property for both the stable and the normal case, the meaning of matrix Q and parameter σ, respectively, differs. In the normal case, Q is the covariance matrix and σ the standard deviation, whereas, in the stable case, Q represents a dispersion matrix and $\sigma = \sqrt{x'Qx}$ the portfolio scale parameter.

As a consequence of the two-fund-separation property for the sub-Gaussian stable model, the market portfolio is equal to the risky tangency portfolio obtained under the equilibrium conditions—as is the case in the conventional mean–variance Capital Asset Pricing Model (CAPM). Therefore, every optimal portfolio amounts to a linear combination of the market portfolio,

$$\overline{x}'r = \frac{r'Q^{-1}(\mu - z_0\mathbf{1})}{\mathbf{1}'Q^{-1}\mu - \mathbf{1}'Q^{-1}\mathbf{1}z_0}, \tag{5.10}$$

and the risk–free return, z_0. As in the mean-variance equilibrium model of Sharpe–Lintner–Mossin, the return of asset i is given by

$$\mathrm{E}(r_i) = z_0 + \beta_{i,m}\left[\mathrm{E}(\overline{x}'r) - z_0\right], \tag{5.11}$$

where $\beta_{im} = \overline{x}' Q e_i / \overline{x}' Q \overline{x}$, with e_i denoting the ith $n \times 1$ unit vector.

From Ross' (1978) necessary and sufficient conditions for two-fund separation we can write

$$r_i = \mu_i + b_i Y + \varepsilon_i, \quad i = 1, ..., n,$$

where $\mu_i = \mathrm{E}(r_i)$, $\varepsilon = [\varepsilon_1, ..., \varepsilon_n]'$, $\mathrm{E}(\varepsilon|Y) = 0$, $b = [b_1, ..., b_n]'$, and vector $bY + \varepsilon$ is sub-Gaussian stable distributed with zero mean. Thus, the sub–Gaussian stable version of the CAPM is not much different from the results in Gamrowski and Rachev (1999) for the stable model, which is a generalization of Fama (1965b) by assuming that $r_i = \mu_i + b_i Y + \varepsilon_i$, for every $i = 1, ..., n$, where ε_i and Y are stable distributed and $\mathrm{E}(\varepsilon|Y) = 0$. Then,

$$\mathrm{E}(r_i) = z_0 + \widetilde{\beta}_{i,m} \left[\mathrm{E}(\overline{x}' r) - z_0 \right],$$

where

$$\widetilde{\beta}_{i,m} = \frac{1}{\alpha \left\| \overline{x}' \widetilde{r} \right\|_\alpha^\alpha} \frac{\partial \left\| \overline{x}' \widetilde{r} \right\|_\alpha^\alpha}{\partial \overline{x}_i} = \frac{[\widetilde{r}_i, \overline{x}' \widetilde{r}]_\alpha}{\left\| \overline{x}' \widetilde{r} \right\|_\alpha^\alpha},$$

can be estimated via (5.3).

For the sub–Gaussian symmetric stable case we have

$$\overline{x}' Q \overline{x} = \left\| \overline{x}' \widetilde{r} \right\|_\alpha^2$$

and

$$\overline{x}' Q e_i = \frac{1}{2} \frac{\partial \left\| \overline{x}' \widetilde{r} \right\|_\alpha^2}{\partial \overline{x}_i},$$

which shows the equivalence of $\beta_{i,m}$ in (5.11) and $\widetilde{\beta}_{i,m}$ in Gamrowski and Rachev (1999), namely,

$$\beta_{i,m} = \frac{\overline{x}' Q e^i}{\overline{x}' Q \overline{x}} = \frac{1}{\sigma_{\overline{x}' r}} \frac{\partial \sigma_{\overline{x}' r}}{\partial \overline{x}_i} = \frac{[\widetilde{r}_i, \overline{x}' \widetilde{r}]_\alpha}{\left\| \overline{x}' \widetilde{r} \right\|_\alpha^\alpha} = \widetilde{\beta}_{i,m},$$

where $\sigma_{\overline{x}' r}$ is the scale parameter of the market portfolio.

2.2 A Three–fund Separation Model in the Domain of Attraction of Stable Laws

The vector $r = [r_1, ..., r_n]'$ gives rise to a three-fund separating stable model, if the return on asset i satisfies

$$r_i = \mu_i + b_i Y + \varepsilon_i, \qquad i = 1, ..., n, \tag{5.12}$$

where the random zero–mean vector $\varepsilon = [\varepsilon_1, \varepsilon_2, ..., \varepsilon_n]'$ is independent from Y and follows a joint sub-Gaussian α_1-stable distribution with $1 < \alpha_1 < 2$ and characteristic function

$$\Phi_\varepsilon(t) = \exp\left(-\left|t'Qt\right|^{\alpha_1/2}\right);$$

and, moreover,

$$Y \stackrel{d}{=} S_{\alpha_2}(\sigma_Y, \beta_Y, 0),$$

that is, Y is α_2-stable distributed random variable with $1 < \alpha_2 < 2$ and zero mean. If $\alpha_1 \neq \alpha_2$, the portfolio returns are in the domain of attraction of an α-stable law with $\alpha = \min(\alpha_1, \alpha_2)$. For $\beta_Y = 0$ and $\alpha_1 = \alpha_2$, the model reduces to the two-fund separation model of Fama (1965b). The characteristic function of the return vector, r, is given by

$$\begin{aligned}\Phi_r(t) =& \Phi_\varepsilon(t)\Phi_Y(t'b)e^{it'\mu} \\ =& \exp\left\{-\left|t'Qt\right|^{\alpha_1/2} - \left|t'b\sigma_Y\right|^{\alpha_2}\left[1 - i\beta_Y \operatorname{sgn}(t'b)\tan\frac{\pi\alpha_2}{2}\right] + it'\mu\right\}'.\end{aligned} \tag{5.13}$$

To estimate (5.12)–(5.13), $\widehat{\mu}$ can, again, be the vector of sample averages; and Y can be a (centered) index return—for example, the market portfolio (5.10) given by the sub-Gaussian model. Note that the random vector ε can be written as a product of a random variable, V, and a Gaussian vector, G, i.e., $\varepsilon = VG$, where $V = \sqrt{A}$, with A being an $\frac{\alpha_1}{2}$-stable subordinator,

$$A \stackrel{d}{=} S_{\alpha_1/2}\left(\left(\cos\frac{\pi\alpha_1}{4}\right)^{2/\alpha_1}, 1, 0\right),$$

and G being a zero mean $n \times 1$ Gaussian vector with covariance matrix Q and independent of A. Given observations $Y^{(k)}, k = 1, \ldots, T$, and letting $\widetilde{r}_i = r_i - \mu_i$ denote the centered returns, Tokat, Rachev and Schwartz (2001) consider the following OLS estimators for $b = [b_1, ..., b_n]'$ and Q:

$$\widehat{b}_i = \frac{\displaystyle\sum_{k=1}^{T} \frac{Y^{(k)}\widetilde{r}_i^{(k)}}{A_k}}{\displaystyle\sum_{k=1}^{T} \frac{\left(Y^{(k)}\right)^2}{A_k}}, \quad i = 1,...,n,$$

and

$$\widehat{Q} = \frac{1}{T}\sum_{k=1}^{T} \frac{1}{A_k}\left(\widetilde{r}^{(k)} - \widehat{b}Y^{(k)}\right)\left(\widetilde{r}^{(k)} - \widehat{b}Y^{(k)}\right)'.$$

Finally, to estimate α_1, one can use the OLS estimator

$$\widetilde{b}_i = \frac{\sum_{k=1}^{T} Y^{(k)} \widetilde{r}_i^{(k)}}{\sum_{k=1}^{T} \left(Y^{(k)}\right)^2}$$

and investigate the fitted residuals, $\widetilde{\varepsilon}^{(k)} = \widetilde{r}^{(k)} - \widetilde{b} Y^{(k)}$. If they are heavy tailed, their tail exponent would be an estimate for α_1.[3]

To determine portfolios that are non-dominated in the sense of Rothschild and Stiglitz (1970),[4] when unlimited short selling is allowed, the scale parameter $\sigma_P = \sqrt{x'Qx}$ needs to be minimized such that $m_P = x'\mu + (1 - x'\mathbf{1})z_0$ and $\widetilde{b} = x'b/\sqrt{x'Qx}$. Ortobelli, Rachev and Schwartz (2000) derive these portfolios by solving the quadratic programming problem

$$\begin{gathered} \min_x x'Qx \quad \text{subject to} \\ x'\mu + (1 - x'\mathbf{1})z_0 = m_P \\ x'b = b^*, \end{gathered} \tag{5.14}$$

for given values of m_P and b^*. In this setting, every portfolio, which maximizes the expected value of a given concave utility function, belongs to the frontier

$$(1 - \lambda_2 - \lambda_3)\, z_0 + \lambda_2 \frac{r'Q^{-1}(\mu - z_0\mathbf{1})}{\mathbf{1}'Q^{-1}(\mu - z_0\mathbf{1})} + \lambda_3 \frac{r'Q^{-1}b}{\mathbf{1}'Q^{-1}b} \tag{5.15}$$

spanned by the riskless return, z_0, and the two risky portfolios

$$P_1 = \frac{r'Q^{-1}(\mu - z_0\mathbf{1})}{\mathbf{1}'Q^{-1}(\mu - z_0\mathbf{1})} \quad \text{and} \quad P_2 = \frac{r'Q^{-1}b}{\mathbf{1}'Q^{-1}b}.$$

Note that, in (5.13), when $\alpha = \alpha_1 = \alpha_2 > 1$, every portfolio, $x'r$, has an $\alpha-$stable distribution and

$$P = (1 - x'\mathbf{1})z_0 + x'r \stackrel{d}{=} S_\alpha(\sigma_{x'r}, \beta_{x'r}, (1 - x'\mathbf{1})z_0 + m_{x'r})$$

for $x \neq 0$, where

$$\sigma_{x'r}^\alpha = \left(x'Qx\right)^{\alpha/2} + \left|x'b\sigma_Y\right|^\alpha,$$

$$\beta_{x'r} = \frac{|x'b\sigma_Y|^\alpha \operatorname{sgn}(x'b)\beta_Y}{\sigma_{x'r}^\alpha},$$

and

$$m_{x'r} = x'\mathrm{E}(r).$$

Hence, this jointly α-stable model is a fund separation model whose solutions satisfy the quadratic programming problem (5.14).

3. Normal and Stable Portfolio Selection: A Comparison

Although there are crucial differences when adopting the stable versus the normal assumption, it is not clear to what extent such a decision matters in practice. To investigate the consequences of relaxing the normal assumption and allowing for heavy–tailed stable return processes, Doganoglu and Mittnik (2002) consider portfolios constructed from a set of 26 stocks belonging to the German DAX index. Using daily observations from January 1991 to April 1998 and assuming a single–index structure, taking the CDAX as index, they compare the consequences of stable versus normal assumptions on portfolios selection and risk assessment. The data reject the hypothesis of joint normality at the 99% level, exhibiting both heavy-tailedness and skewness.

Adopting a set–up as specified in (5.12) by allowing for asymmetry, i.e., $\beta_Y \neq 0$, but restricting $\alpha_1 = \alpha_2 = \alpha$, one obtains 1.706 as the maximum–likelihood estimate for the overall stable index if a symmetry restriction is imposed (i.e., $\beta_Y = 0$) and 1.705 if β_Y is unrestricted; then, its estimate is $\hat{\beta} = 0.121$. Likelihood–ratio tests clearly favor the heavy–tailed stable model over the normal, and the symmetric stable is rejected in favor of the asymmetric version.

To assess the implications of the distributional assumption on the optimal portfolio weights, one can construct an efficient frontier in the expected return–risk space by maximizing the expected portfolio return for a given risk level. The difficulty here is that the usual risk measure, σ, for the normal and the stable are not comparable, given that the α-values differ. However, when measuring risk in terms of VaR, valid comparisons can be made. The results for the DAX portfolio indicate that the optimal portfolios for given VaR levels differ considerably with respect to the portfolio weights, when assuming normal versus and stable returns. Moreover, it turns out that the distributional assumption severely affects the risk assessment for the optimal portfolios. Table 5.1 illustrates this for three different target returns. It shows the estimated VaR values as implied by the normal, symmetric and asymmetric stable models and compares them to the empirical VaR values, considering five different target probabilities, namely the 10%, 7.5%, 5%, 2.5%, and the 1% level. (In the following we write, for example, $\text{VaR}_{.05}$ for the VaR value at 5% target probability.)

The results in Table 5.1 demonstrate that risk assessment based on the commonly employed VaR measure is highly sensitive with respect to the adopted distributional assumption as far as the tail index α is concerned. A relaxation of the symmetry assumption for the stable model has only marginal impact on the VaR estimate when the target probabilities are large. However, further out in the tails, it makes much more of a difference when allowing for asymmetry. Overall, the stable VaR estimates are much closer to empirical VaR values, whereas the normal estimates exhibit systematic deviations. They consistently overes-

Table 5.1. Estimated VaR Values for Different Target Returns

	Daily Target Return (%)		
	0.10	0.15	0.20
		$\text{VaR}_{.10}$	
Empirical Value	0.987	1.266	1.834
Normal Fit	1.143	1.548	2.284
Sym. Stable Fit	1.040	1.358	1.884
Asym. Stable Fit	1.041	1.358	1.882
		$\text{VaR}_{.075}$	
Empirical Value	1.204	1.529	2.150
Normal Fit	1.296	1.757	2.590
Sym. Stable Fit	1.210	1.583	2.195
Asym. Stable Fit	1.203	1.572	2.178
		$\text{VaR}_{.05}$	
Empirical Value	1.444	1.840	2.568
Normal Fit	1.495	2.029	2.988
Sym. Stable Fit	1.458	1.911	2.648
Asym. Stable Fit	1.436	1.881	2.603
		$\text{VaR}_{.025}$	
Empirical Value	1.990	2.545	3.372
Normal Fit	1.801	2.447	3.599
Sym. Stable Fit	1.944	2.555	3.537
Asym. Stable Fit	1.886	2.475	3.423
		$\text{VaR}_{.01}$	
Empirical Value	2.766	3.166	4.674
Normal Fit	2.155	2.931	4.308
Sym. Stable Fit	2.912	3.835	5.306
Asym. Stable Fit	2.764	3.636	5.027

timate the portfolios' riskiness for larger target probabilities and underestimate the risk for the 2.5% and the 1% level.

Table 5.2 compares the percentage errors of the VaR estimates from the normal and the asymmetric stable model. With one exception, the (absolute) relative error of the normal always exceeds that of the asymmetric stable.

The normal model's underestimation of the portfolio risk at the 1% target probability is highly undesirable for banks, using an internal model to assess the market risk of their trading positions. Banks have to continually monitor the adequacy of the internal model by performing certain "backtesting" procedures. According to the Basle Accord, national oversight authorities ought to judge the adequacy of an internal model by evaluating the number of violations of the $\text{VaR}_{.01}$–limit for a one–day holding period during the preceding 250–day period. A correctly specified model should, on average, produce 2.5 violations

Table 5.2. Relative Errors of VaR Estimates

	Daily Target returns (%)		
	0.10	0.15	0.20
	$\text{VaR}_{.10}$–Error (%)		
Normal Fit	16	22	25
Asym. Stable Fit	5	7	3
	$\text{VaR}_{.075}$–Error (%)		
Normal Fit	8	15	20
Asym. Stable Fit	0	3	1
	$\text{VaR}_{.05}$–Error (%)		
Normal Fit	4	10	16
Asym. Stable Fit	-1	3	1
	$\text{VaR}_{.025}$–Error (%)		
Normal Fit	-9	-4	7
Asym. Stable Fit	-5	-3	2
	$\text{VaR}_{.01}$–Error (%)		
Normal Fit	-22	-7	-8
Asym. Stable Fit	0	15	8

of the $\text{VaR}_{.01}$–limit over this period. If an excessive number of such violations occurs, banks may be forced to increase their capital requirements; they may even be disallowed from using the model.

The Basle Committee has established specific standards for evaluating the adequacy of an internal model by defining three "zones" for the violation frequency for the $\text{VaR}_{.01}$–limit over a 250–day horizon:

Green zone (0–4 violations): The model is considered to be accurate and no supervisory response is required.

Yellow zone (5–9 violations): The model is more likely to be inaccurate than accurate. Capital requirements should be increased by raising the multiplication factor from 3 to somewhere between 3.4 and 3.85.

Red zone (10 or more violations): There is a presumption that the model is flawed. The bank's capital requirements should be automatically increased by raising the multiplication factor from 3 to 4. Moreover, the bank should be required to immediately work on improving the model.

The specific increments, with which the multiplication factor ought to be raised in case of yellow, depend on the violation frequency. In fact, the guidelines state that "[t]he concern about 'fat tails' was also an important factor in the choice of the specific increments..." (Basle Committee, 1996, p. 8).

To examine how the normal and stable models fair under the Basle Committee's adequacy standards, Doganoglu and Mittnik (2002) perform backtesting

Table 5.3. Backtesting Results for Seven Non–overlapping Subsamples and Four $\mathrm{VaR}_{.01}$–Limits

	Number of Cases (out of 28)	
	Normal	Stable
Green Zone	15	26
Yellow Zone	12	2
Red Zone	1	0

exercises using the three–zone approach. Dividing the sample into seven non-overlapping subsamples, comprised of 250 days each and considering four different $\mathrm{VaR}_{.01}$–limit targets, there are altogether 28 cases for which violation frequencies can be compared. The results of this comparison, summarized in Table 5.3, are rather disastrous for the normal model. Only in 15 out of the 28 cases does the normal model end up in the green zone; it falls 12 times into the yellow, and in one case even into the red zone. The stable model performs consideraby better in this backtesting comparison. Only in two cases does it fall into the yellow zone and never into the red.

4. Conclusions

We have seen that the sub-Gaussian stable assumption for asset returns permits a mean–risk analysis that is rather similar to the mean–variance framework of Markowitz. In fact, it admits the same analytical form for the efficient frontier, but the interpretation of the parameters of the two models differs. For practical applications, the most important difference arises from the required estimation procedures. Heavy–tailed asymmetric returns, give rise to a three–fund separation model, where the portfolios are in the domain of attraction of an (α_1, α_2)–stable law.

An empirical comparison of the stable with the normal approach to portfolio selection reveals that the stable model produces different optimal portfolio weights and, moreover, provides more accurate assessments of portfolio risk.

The empirical findings suggest that it can be very costly for banks to ignore the asset returns' heavy–tailedness and by simply relying on the normal assumption.

Notes

1. Several monographs are devoted to the stable Paretian modeling. Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994) provide detailed accounts of theoretical aspects of stable distributed random variables; Rachev and Mittnik (2000) consider stable Paretian models in financial applications.

2. Note that for every $x \in \mathbb{R}^n$, we have $x'\widehat{Q}x > 0$ if and only if $\widehat{Q} + \widehat{Q}'$ is a positive definite matrix. This can be checked to avoid negative scale parameter estimates.

3. The asymptotic properties of the above estimator can be derived following the arguments in Paulauskas and Rachev (1999) and Götzenberger, Rachev and Schwartz (2000).

4. See also Ortobelli, Huber, Rachev, and Schwartz (2001).

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